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On Constrained Motion.

BY PETER FIELD.

Introduction.

The traditional problem in the mechanics of a particle supposes given a system of particles with a certain number of degrees of freedom and a system of forces; the solution of the problem consists in determining the motion of each particle. Lagrange's equations enable us to do this provided it is possible to overcome the difficulties incident to the integration of the equations of motion.

Painlevé* has shown that in discussing problems of this kind, if we assume Coulomb's laws of friction to hold true, we must proceed with caution, as the supposition that motion takes place may lead either to conditions which are incompatible or to the possibility of more than one solution.† In some of the problems discussed by Painlevé, our interest is not primarily in the whole history of the subsequent motion but rather in knowing how many values of the acceleration are possible at the given moment with the given initial conditions. This is the point of view which is adopted in this paper.

Efforts have been made to dispose of the cases where more than one motion is possible by taking account of the elasticity of the material and also by supposing that it must be possible for the frictional force to increase from zero up to its maximum value,‡ but the fact nevertheless remains that when we suppose that we are dealing with a rigid system and suppose the frictional force at any point is proportional to the normal pressure at that point, and make no additional assumptions, we are led to cases where there may be more than one solution.§

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* "Leçons sur le frottement."

† See my paper on Coulomb's laws of friction, *Zeitschrift für Mathematik und Physik,* Vol. LXI, p. 68.

‡ See Lecornu, *Comptes rendus,* Vol. CXL (1905), p. 635, and De Sparre, Vol. CXLI (1905), p. 310; also *Zeitschrift für Mathematik und Physik,* Vol. LVIII (1910), articles by Klein, Mises, Hamel, Prandtl, Pfeiffer.

§ See Appell, *Traité de mécanique rationnelle,* Vol. II, p. 127, 3d edition.

Statement of the Problem.

The purpose of this paper is to investigate what conditions must be satisfied in order that the supposition that motion takes place may lead to conditions which are compatible, when there are no external forces and we have two particles m_1 and m_2 connected by a weightless rod of length l and one or both are constrained. The different cases may be tabulated as follows: *

(a) m_1 is constrained to a surface and m_2 is free,

(b) m_1 is constrained to a curve and m_2 is free,

(c) m_1 and m_2 are both constrained to surfaces,

(d) m_1 is constrained to a surface, m_2 to a curve,

(e) m_1 and m_2 are both constrained to curves.

The data are chosen as follows: The particle of mass m_1 is at P_1 (Fig. 1) and the second particle is at P_2. The velocities of m_1 and m_2 are u_1 and u_2.

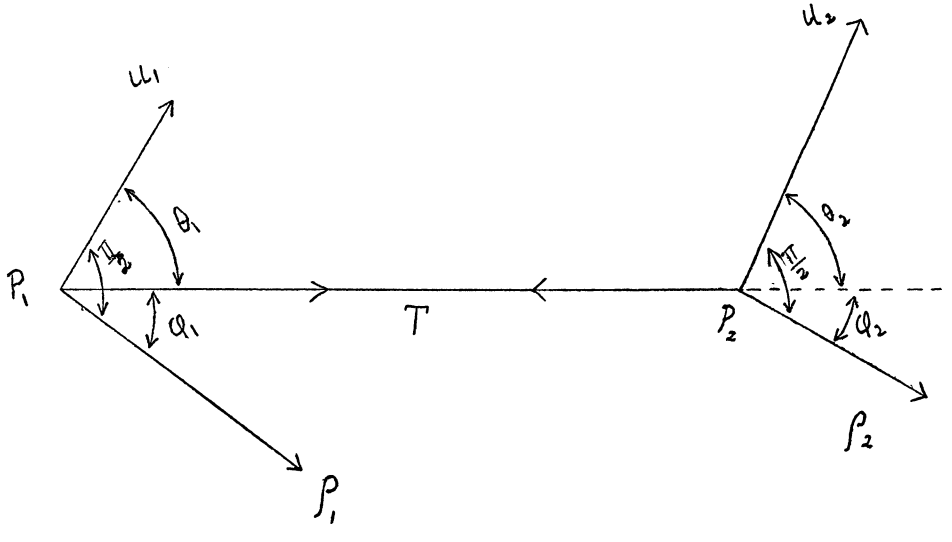

Fig. 1.

The projections of these velocities along l and at right angles to l are (v, v_1), (v, v_2) respectively, the projections along l being equal as P_1P_2 is of constant length. The angle between v_1 and v_2 will be called θ. In case the particles are constrained to curves, ρ_1 and ρ_2 are the radii of curvature of the paths; while if the particles are constrained to surfaces, ρ_1 and ρ_2 are the radii of curvature of the plane sections determined by the velocity and the normal to the surface. The coefficients of friction will be denoted by μ_1 and μ_2.

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* Some special cases have been studied by Painlevé and others. A study of the case where the two particles are constrained to parallel plane curves will appear in one of the forthcoming numbers of the *Zeitschrift für Mathematik und Physik.*

The angles θ_1 and θ_2 shown in Fig. 1 can be expressed in terms of v, v_1, v_2; but ϕ_1 and ϕ_2, also shown in the figure, are independent of the preceding parameters. We therefore have the following thirteen independent parameters: $m_1, m_2, \mu_1, \mu_2, l, v, v_1, v_2, \phi_1, \phi_2, \theta, \rho_1, \rho_2$. The tension in the rod is called T.

Case (a): *m_1 is Constrained to a Surface and m_2 is Free.*

The projections of the acceleration of m_1 are $\frac{u_1^2}{\rho_1}$ along ρ_1, $\frac{du_1}{dt}=j_1$ in the direction of u_1, and $\frac{T}{m_1}(1-\cos^2\phi_1-\cos^2\theta_1)^{\frac{1}{2}}$ along a line perpendicular to u_1 and ρ_1 (Fig. 2).

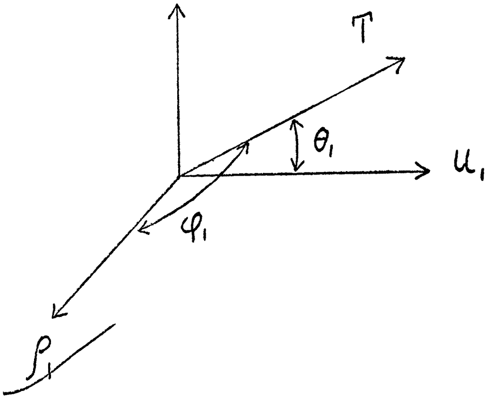

FIG. 2.

In order to get the value of j_1 it is necessary to find the pressure against the surface. This pressure is

$$\left|-m_1\frac{v^2+v_1^2}{\rho_1}+T\cos\phi_1\right|,$$

and we have at once the equation

$$m_1 j_1=-\mu_1\left|-m_1\frac{v^2+v_1^2}{\rho_1}+T\cos\phi_1\right|+T\cos\theta_1.$$

The acceleration of m_2 is directed along P_1P_2 and is equal to $-\frac{T}{m_2}$. Now as the distance between the two particles is constant, the projections of their accelerations along P_1P_2 differ by

$$\omega^2 l = \frac{v_1^2 + v_2^2 - 2v_1v_2\cos\theta}{l},$$

ω being the angular velocity of l and consequently equal to the geometric sum of v_1 and $-v_2$ divided by l.

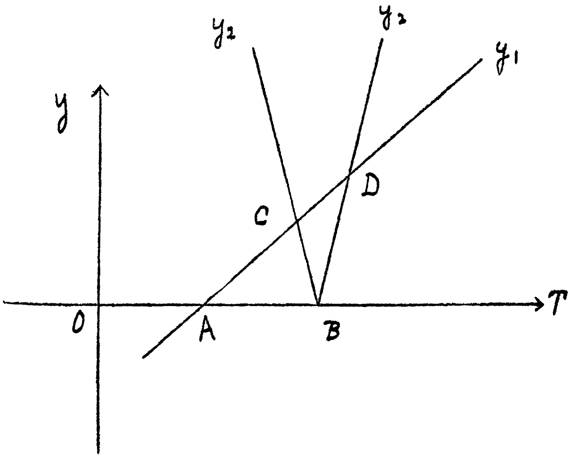

FIG. 3.

In case motion is possible we must therefore be able to determine T so as to satisfy the equation

$$\frac{T}{m_1}\cos^2\theta_1 - \mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1 - \frac{u_1^2}{\rho_1}\right| + \frac{u_1^2}{\rho_1}\cos\phi_1 + \frac{T}{m_1}(1-\cos^2\phi_1-\cos^2\theta_1) - \frac{v_1^2+v_2^2-2v_1v_2\cos\theta}{l} + \frac{T}{m_2} = 0.$$

This equation may be rendered a trifle more compact by substituting k^2 and k_1^2 for $\frac{v_1^2+v_2^2-2v_1v_2\cos\theta}{l}$ and $\frac{u_1^2}{\rho_1}$. It can then be written

$$T\left(\frac{\sin^2\phi_1}{m_1} + \frac{1}{m_2}\right) + k_1^2\cos\phi_1 - k^2 = \mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1 - k_1^2\right|.$$

This condition is easily interpreted geometrically. In Fig. 3 the values of

$$y_1 = T\left(\frac{\sin^2\phi_1}{m_1} + \frac{1}{m_2}\right) + k_1^2\cos\phi_1 - k^2$$

are represented by points on a straight line, while the values of

$$y_2=\mu_1 \cos \theta_1 \left| \frac{T}{m_1} \cos \phi_1 - k_1^2 \right|$$

are represented by points on a broken line which will lie above or below the T axis according as θ_1 is acute or obtuse. The values of T at the points C and D satisfy the conditions of our problem. If the data were so chosen that A coincided with B and the slopes of y_1 and one segment of the broken line were equal, we should have an infinite number of solutions. If θ_1 is acute and A lies to the left of B, there will be but one solution if the slope of the line y_1 is greater than the slope of the broken line; while if A lies to the right of B and the slope of y_1 is less than the slope of the broken line, our equation has no solution. The problem may therefore have 0, 1, 2, or an infinite number of solutions depending on how the data are chosen.

Case (b): *m_1 is Constrained to a Curve and m_2 is Free.*

The acceleration of m_1 lies in the osculating plane of the twisted curve to which m_1 is constrained. The component along the radius of curvature has the same value as in case (a), but the value of j_1 is different, as the pressure against the curve is now the geometric sum of $-m_1 \frac{u_1^2}{\rho_1}$ and the projection of T on the normal plane to the curve at P_1. This pressure (Fig. 2) is equal to

$$\sqrt{T^2 \sin^2 \theta_1 - 2Tm_1 \frac{u_1^2}{\rho_1} \cos \phi_1 + m_1^2 \frac{u_1^4}{\rho_1^2}}.$$

Therefore

$$j_1=\frac{T}{m_1} \cos \theta_1 - \mu_1 \sqrt{\frac{T^2}{m_1^2} \sin^2 \theta_1 - 2\frac{T}{m_1} \frac{u_1^2}{\rho_1} \cos \phi_1 + \frac{u_1^4}{\rho_1^2}}.$$

If we now write the condition that the projection of the acceleration of m_1 along the rod is equal to $-\frac{T}{m_2}+\omega^2 l$ and use the same notation as in (a), we have

$$T\left(\frac{\cos^2 \theta_1}{m_1}+\frac{1}{m_2}\right)+k_1^2 \cos \phi_1 - k^2 = \mu_1 \cos \theta_1 \sqrt{\frac{T^2}{m_1^2} \sin^2 \theta_1 - 2\frac{T}{m_1} k_1^2 \cos \phi_1 + k_1^4}$$

as the condition which T must satisfy.

The possible number of solutions is the same as in case (a); *i. e.*, it may be 0, 1, 2, or an infinite number. Instead of having the points of intersection of a straight line and a broken line as in Fig. 3, we now have a straight line and an arc of an hyperbola. In order to have an infinite number of solutions it is necessary that the hyperbola degenerate. In that case $\cos\phi_1 = \pm\sin\theta_1$, which means that the radius of curvature lies in the plane of the connecting rod and the direction of the velocity of m_1. This condition alone is of course not sufficient to give an infinite number of solutions. In addition two conditions corresponding to those of case (a) must be satisfied.

There is no difficulty in satisfying these conditions. For instance take $\cos\phi_1 = \sin\theta_1$ and suppose θ_1 acute. The other two conditions are

$$\frac{m_1 k_1^2}{\sin\theta_1} = \frac{k^2 - k_1^2 \sin\theta_1}{\dfrac{\cos^2\theta_1}{m_1} + \dfrac{1}{m_2}},$$

$$\frac{\cos^2\theta_1}{m_1} + \frac{1}{m_2} = \frac{\mu_1 \cos\theta_1 \sin\theta_1}{m_1}.$$

The first might be regarded as an equation to determine k^2 and the second as an equation which determines μ_1.

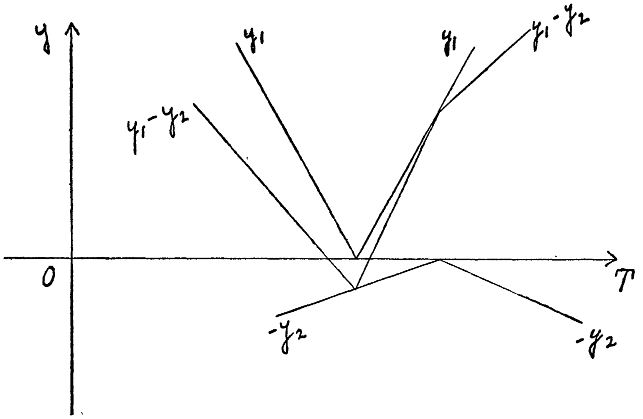

FIG. 4.

Case (c): *m_1 and m_2 are Both Constrained to Surfaces.*

According to (a) the projection of the acceleration of m_1 along P_1P_2 is

$$\frac{T}{m_1}\sin^2\phi_1 + k_1^2\cos\phi_1 - \mu_1\cos\theta_1 \left| \frac{T}{m_1}\cos\phi_1 - k_1^2 \right|.$$

As the projection of the acceleration of m_2 along this line has the same form provided we replace T by $-T$, θ_1 by θ_2, ϕ_1 by ϕ_2, and k_1^2 by $\dfrac{u_2^2}{\rho_2} = k_2^2$, the equation which determines T may be written

$$T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\sin^2\phi_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2$$
$$=\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|-\mu_2\cos\theta_2\left|\frac{T}{m_2}\cos\phi_2+k_2^2\right|,$$

or $y=y_1-y_2$. It is no restriction to suppose the angles θ_1 and θ_2 acute. The values of

$$y_1-y_2=\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|-\mu_2\cos\theta_2\left|\frac{T}{m_2}\cos\phi_2+k_2^2\right|$$

will then lie on a broken line composed of three rectilinear segments as in Fig. 4.

The desired values of T are found as the values of T at the points of intersection of the broken line with the line

$$y=T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\sin^2\phi_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2.$$

This gives rise to 0, 1, 2, 3, or an infinite number of solutions, depending on how the values of the constants are chosen. Some of the constants, as μ_1, μ_2, m_1, m_2, k_1^2, k_2^2, k^2, are from their nature positive, but even with this restriction cases with the different numbers of solutions are readily constructed.

As an illustration let us build a case with an infinite number of solutions We might take the vertices of the broken lines at the same point and have the line

$$y=T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\sin^2\phi_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2$$

coincide with that part of the broken line y_1-y_2 which lies to the right of the common vertex of y_1 and y_2 (Fig. 5). This can readily be accomplished by taking

$$m_1=m_2=1,\qquad \cos\phi_2=-\cos\phi_1,$$

$$\mu_1=\frac{\mu_2\cos\theta_2\cos\phi_1+2\sin^2\phi_1}{\cos\theta_1\cos\phi_1},$$

$$k^2=2k_1^2\frac{\cos^2\phi_1+\sin^2\phi_1}{\cos\phi_1}=\frac{2k_1^2}{\cos\phi_1};$$

μ_1 and k^2 being positive, we add the further restrictions that θ_1, θ_2, ϕ_1 are acute.

Case (d): *m_1 is Constrained to a Surface and m_2 to a Curve.*

From (a) and (b) the projection of the acceleration of m_1 along the rod is

$$T\frac{\sin^2\phi_1}{m_1}+k_1^2\cos\phi_1-\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|$$

when m_1 is constrained to a surface and

$$T\frac{\cos^2\theta_1}{m_1}+k_1^2\cos\phi_1-\mu_1\cos\theta_1\sqrt{\frac{T^2}{m_1^2}\sin^2\theta_1-2\frac{T}{m_1}k_1^2\cos\phi_1+k_1^4}$$

when constrained to a curve. If m_2 is constrained to a curve, the value of the projection of its acceleration along the rod is obtained from the corresponding expression for m_1 by replacing T by $-T$ and making the subscripts 2 in place of 1. From this it follows at once that when m_1 is constrained to a surface and m_2 to a curve, the stress in the rod must satisfy the equation

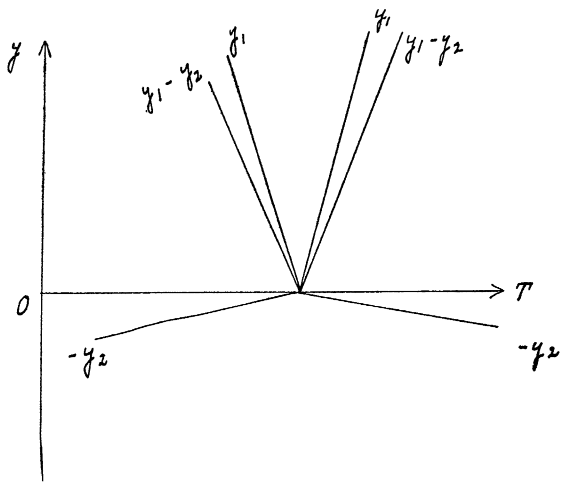

FIG. 5.

$$T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\cos^2\theta_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2$$
$$=\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|-\mu_2\cos\theta_2\sqrt{\frac{T^2}{m_2^2}\sin^2\theta_2+2\frac{T}{m_2}k_2^2\cos\phi_2+k_2^4},$$

or more briefly, $y=y_1-y_2$.

Suppose the angles θ_1 and θ_2 are obtuse; the values of y_1-y_2 corresponding to the different values of T will then lie on two arcs of hyperbolas as illustrated in Fig. 6. The values of T which satisfy our problem are found as the values of T at the points where the curve y_1-y_2 is cut by the line

$$y=T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\cos^2\theta_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2.$$

It is easily shown that there are 0, 1, 2, 3, 4 or an infinite number of solutions depending on how the data of the problem are chosen.

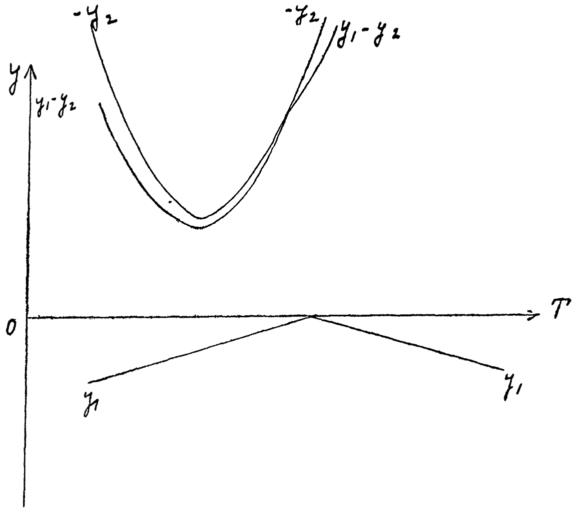

FIG. 6.

Case (e): *m_1 and m_2 are Both Constrained to Curves.*

By making use of the results already derived we have at once the equation for T; *viz.*,

$$T\left(\frac{\cos^2\theta_1}{m_1}+\frac{\cos^2\theta_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2$$

$$=\mu_1\cos\theta_1\sqrt{\frac{T^2}{m_1^2}\sin^2\theta_1-2\frac{T}{m_1}k_1^2\cos\phi_1+k_1^4}$$

$$-\mu_2\cos\theta_2\sqrt{\frac{T^2}{m_2^2}\sin^2\theta_2+2\frac{T}{m_2}k_2^2\cos\phi_2+k_2^4},$$

or $y=y_1-y_2$.

The values of y_1-y_2 corresponding to different values of T will lie on an arc of a quartic curve, and we have 0, 1, 2, 3, 4 or an infinite number of solutions for our problem, just as in case (d).

An Elastic Connecting Rod.

Although the case where the connecting rod is elastic does not come within the scope of this paper, as we are dealing with a rigid system, it may be worth mentioning that it is at once apparent that when the rod is supposed elastic, the stress can have but one value, which is determined by the strain, and consequently the accelerations at P_1 and P_2 have but a single value.

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